Weak and variational formulations for BIEs related to the wave equation

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Abstract. In this paper we consider a Dirichlet or Neumann problem of one-dimensional wave propagation reformulated using space-time boundary integral equations (BIE) with retarded potential. In the first part, starting from a natural energy identity, a space-time weak formulation for the BIEs is presented and continuity and coerciveness properties of the related bilinear form are proved. Then, following general techniques suggested in [2, 22], definition of suitable functionals whose minimum (or saddle) point is the solution of the given BIE and extended variational formulations in the spirit of [7] are proposed. Some numerical results, obtained using different approximation techniques, will be presented and discussed.

Keywords: Boundary integral equations (BIE), boundary element method (BEM), one-dimensional wave propagation, weak formulation, (extended) variational formulation.

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1 Introduction

Time-dependent problems that are frequently modelled by hyperbolic partial differential equations can be dealt with the boundary integral equations method. When we have a homogeneous partial differential equation with constant coefficients, the initial conditions vanish, the data are given only on the boundary of the domain and this does not depend on time, the transformation of the problem to a boundary integral equation follows the same well-known method for elliptic boundary value problems. Boundary element methods (BEM) have been successfully applied to many such problems from fields as electromagnetic wave propagation, computation of transient acoustic wave, linear elastodynamics, fluid dynamics, etc. [1, 3, 4, 10, 12, 13, 20]. Frequently claimed advantages over domain approaches are the dimensionality reduction, the easy implicit enforcement for radiation conditions at infinity, reduction of an unbounded exterior domain to a bounded boundary, the high accuracy achievable and simple pre- and post-processing for input and output data.

In principle, both the frequency-domain and time-domain BEM can be used. Most earlier contributions concerned direct formulations of BEM in the frequency domain, often using the Laplace or Fourier transforms and addressing wave propagation problems. After this transformation a standard boundary integral method for an elliptic problem (Helmoltz problem) is applied and then the transformation back to time domain employs special methods for the inversion of Laplace or Fourier transforms [17, 18]. In this direction the most interesting results are given by the weak formulation due to Bamberger and Ha Duong [5, 6]. The time-domain BEM yields directly the unknown time-dependent quantities. In this last approach, the construction of the boundary integral equations, via representation formula in terms of single layer and double potentials, uses the fundamental solution of the hyperbolic partial differential equation and jump relations. Causality condition and time-invariance

imply that the integral equations are of Volterra type in time variable and of convolution type in time, respectively.
Here we consider a Dirichlet or Neumann problem for a temporally homogeneous one-dimensional wave equation, reformulated as a boundary integral equation with retarded potential, but only the space-time BEM will be used. Special attention is devoted to a formulation based on a natural energy identity that leads to a space-time weak formulation of the corresponding BIE with robust theoretical properties. Continuity and coerciveness of the bilinear form related to this last formulation are proved in this paper. We believe that this formulation is an effective alternative to that proposed in [5] and obtained from a direct Laplace transform of the differential problem.
The second part of this paper is dedicated to an introduction of (extended) variational formulations in the spirit of [2, 7, 22], which allow the definition of suitable functionals whose minimum (or saddle) point is solution of the given BIE. In fact, variational formulations can be used as a starting point for the construction of numerical solution methods. Nevertheless, it is not always easy to give a variational formulation to a mathematical problem. Tonti [22] has proposed a general technique to obtain variational formulation associated with any nonlinear problem; a similar method with reference to linear non-potential operators has been proposed in [19, 21]. Few years later, Auchmuty [2] developed Tonti's ideas. Independently of the last work, Carini [8] and Carini and Genna [9] have improved this technique with reference to several specific applications in the field of continuum mechanics.
Numerical results comparing different approximation techniques for one-dimensional wave propagation problems will be presented and discussed. Instabilities phenomena, which arise from a classical L^2 or convolutive weak formulations of the corresponding BIEs, could be prevented using suitable time steps in the discretization phase or solving carefully the final algebraic problem, of course with a higher computational cost with respect to the energetic procedure, which appears to be unconditionally stable.

2 One dimensional wave equation

2.1 Dirichlet problem

In this section we consider the retarded potential representation of solutions of the Dirichlet problem for a one dimensional wave equation, from which we will deduce a suitable boundary integral reformulation of the differential problem.
Let $\Omega = (0, L) \subset \mathbb{R}$ and let $u(x, t)$ be the solution to the wave problem

$$u_{tt} - u_{xx} = 0, \qquad x \in \mathbb{R} \setminus \{0, L\}, \quad t \in (0, T), \tag{1}$$

$$u(x, 0) = u_t(x, 0) = 0, \qquad x \in \mathbb{R} \setminus \{0, L\}, \tag{2}$$

$$u(x, t) = g(x, t), \qquad (x, t) \in \Sigma_T := \{0, L\} \times [0, T], \tag{3}$$

where $g(x, t)$ is a given function. Note that u is considered as the solution on the whole $\mathbb{R}$, not only in Ω. Whenever necessary we shall distinguish the internal solution u^-, i.e. for $x \in \Omega$, from the external one u^+, i.e. for $x \in \mathbb{R} \setminus \Omega$.
In order to rewrite problem (1)-(3) as a boundary integral equation, we need to recall the expression of the forward fundamental solution $G(x, t)$ of the wave operator:

$$G(x, t) = \frac{1}{2} H(t - |x|) = \frac{1}{2} H(t) \left(H(x + t) - H(x - t) \right), \tag{4}$$

where $H(t)$ is the Heaviside step function. Using the fundamental solution (4) we obtain the single layer representation formula of the solutions of the equation (1) for $t \in \mathbb{R}$, $x \in \mathbb{R} \setminus \{0, L\}$:

$$
\begin{aligned}
u(x,t) = (V\varphi)(x,t) &= \int_{-\infty}^{+\infty} G(x,t-\tau)\varphi(0,\tau)d\tau + \int_{-\infty}^{+\infty} G(x-L,t-\tau)\varphi(L,\tau)d\tau \\
&= \frac{1}{2}\int_{-\infty}^{t} H(t-\tau-|x|)\varphi(0,\tau)d\tau + \frac{1}{2}\int_{-\infty}^{t} H(t-\tau-|x-L|)\varphi(L,\tau)d\tau \\
&= \frac{1}{2}\int_{-\infty}^{t-|x|} \varphi(0,\tau)d\tau + \frac{1}{2}\int_{-\infty}^{t-|x-L|} \varphi(L,\tau)d\tau, \qquad (5)
\end{aligned}
$$

where

$$
\varphi = \left[\frac{\partial u}{\partial \nu}\right] := \frac{\partial u^-}{\partial \nu} - \frac{\partial u^+}{\partial \nu} \qquad (6)
$$

is the jump of the normal derivatives of u at $x = 0$ and $x = L$, and ν is fixed as the unitary outward normal with respect to the boundary of Ω.
Since problem (1)-(3) is formulated on the time interval $[0,T]$, in order to keep our notations as simple as possible, hereafter we shall consider functions φ defined on the whole real line but having support only in the fixed time interval $[0,T]$. From formula (5), taking the limits as $x \to 0$ and $x \to L$, we obtain the following system of boundary integral equations for the unknown φ at the endpoints of the interval $(0,L)$:

$$
\left.
\begin{aligned}
(V\varphi)(0,t) &= \frac{1}{2}\Big[\int_0^{t} \varphi(0,\tau)d\tau + \int_0^{t-L} \varphi(L,\tau)d\tau\Big] = g(0,t) \\
(V\varphi)(L,t) &= \frac{1}{2}\Big[\int_0^{t-L} \varphi(0,\tau)d\tau + \int_0^{t} \varphi(L,\tau)d\tau\Big] = g(L,t)
\end{aligned}
\right\} \quad t \in [0,T], \qquad (7)
$$

that can be written with the compact notation

$$
V\varphi = g\,, \qquad (8)
$$

where $\varphi(t)$, $g(t)$ denote the vector valued functions $(\varphi(0,t),\varphi(L,t))^\top$ and $(g(0,t),g(L,t))^\top$, respectively. In order to formulate the operator equation (8) in a suitable functional framework, we assume V as defined in $L^2(\Sigma_T)$. With this choice, from (7) it easy to verify that the range of V lies in $H^1_{\{0\}}(\Sigma_T)$, the space of $H^1(\Sigma_T)$ functions vanishing for $t = 0$. Hence hereafter we shall consider

$$
V : L^2(\Sigma_T) \to H^1_{\{0\}}(\Sigma_T).
$$

As a consequence of the next subsection results, V turns out to be an isomorphism between these two Hilbert spaces.

2.2 Energetic weak problem related to the BIE $V\varphi = g$

A classical way to introduce a weak formulation for (8) is to project the BIE using $L^2(\Sigma_T)$ scalar product. Now, considering the bilinear form $a_{L^2}(\varphi,\psi) : L^2(\Sigma_T) \times L^2(\Sigma_T) \to \mathbb{R}$ defined by

$$
a_{L^2}(\varphi,\psi) := < V\varphi, \psi >_{L^2(\Sigma_T)} = \int_0^T \psi(0,t)\, V\varphi(0,t)dt + \int_0^T \psi(L,t)\, V\varphi(L,t)dt\,, \qquad (9)
$$

we can write the following weak problem:

given $g \in H^1_{\{0\}}(\Sigma_T)$, find $\varphi \in L^2(\Sigma_T)$ such that

$$
a_{L^2}(\varphi,\psi) = < g, \psi >_{L^2(\Sigma_T)}, \quad \forall\, \psi \in L^2(\Sigma_T). \qquad (10)
$$

There are two major drawbacks in the above formulation: the bilinear form $a_{L^2}(\cdot,\cdot)$ is not coercive, in fact choosing $\varphi \equiv \psi$, the formula (10) does not give a positive definite expression; further, it is implicit in the weak formulation (10) that the equation (8) must be understood considering the compact operator $V : L^2(\Sigma_T) \to L^2(\Sigma_T)$, which obviously cannot have continuous inverse. As a consequence, it is not surprising that problem (10) gives rise to instability phenomena in the discretization phase, as it will be shown in Section 5.
An alternative approach is suggested by the well-known conservation law satisfied by the solutions to the D'Alembert equation:

$$0 = u_t(u_{tt} - u_{xx}) = \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right) - \frac{\partial}{\partial x}(u_t u_x)\,.$$

Integrating with respect to space-time in $\mathbb{R} \times (0,T)$ and taking into account that u and u_t vanish for $t = 0$, we get the energy identity

$$\mathcal{E}(T) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_t^2 + u_x^2)\,dx\,\Big|_{t=T} = \int_0^T u_t \cdot \left[\frac{\partial u}{\partial \nu}\right] dt = \int_0^T (V\varphi)_t \cdot \varphi\,dt\,. \tag{11}$$

The quadratic form appearing in the last term of (11) leads to a natural space-time weak formulation of the corresponding boundary integral equation (8) with robust theoretical properties. In fact, the advantages of this approach are the following:
(i) we can localize the problem to a finite time interval $[0,T]$;
(ii) the quadratic form given by the energy, i.e.

$$\mathcal{E}(T) =< (V\varphi)_t, \varphi >_{L^2(\Sigma_T)}\,,$$

is, at least in the one dimensional case, both continuous and coercive in the appropriate spaces, i.e. exactly the functional spaces where the Dirichlet problem is well-posed [15].

In order to derive continuity and coerciveness properties of the total energy $\mathcal{E}(T)$, we concentrate our analysis on the operator $A:\ L^2(\Sigma_T) \to L^2(\Sigma_T)$, defined as

$$A\varphi(t) := (V\varphi)_t(t) = \begin{bmatrix} (V\varphi)_t(0,t) \\ \\ (V\varphi)_t(L,t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \varphi(0,t) + H(t-L)\varphi(L,t-L) \\ \\ H(t-L)\varphi(0,t-L) + \varphi(L,t) \end{bmatrix},\ t \in [0,T]. \tag{12}$$

By an application of the Cauchy-Schwarz inequality, we have immediately that A is a continuous operator. We state this property in the following

Proposition 1 *For any given time T, the operator $A : L^2(\Sigma_T) \to L^2(\Sigma_T)$, defined in (12) is bounded, with norm $\|A\| \le 1$.*

More interesting are the positivity properties of the quadratic form associated to the operator A. Having introduced the bilinear form $a_{\mathcal{E}}(\varphi,\psi) : L^2(\Sigma_T) \times L^2(\Sigma_T) \to \mathbb{R}$ defined by

$$a_{\mathcal{E}}(\varphi,\psi) :=< A\varphi, \psi >_{L^2(\Sigma_T)}= \int_0^T \psi(0,t)\,(V\varphi)_t(0,t)\,dt + \int_0^T \psi(L,t)\,(V\varphi)_t(L,t)\,dt\,, \tag{13}$$

we have the following

Theorem 1 *For every $T > 0$ there exists a positive constant $c(T)$ such that*

$$a_{\mathcal{E}}(\varphi,\varphi) \ge c(T)\ |\varphi|^2_{L^2(\Sigma_T)}, \qquad \varphi \in L^2(\Sigma_T). \tag{14}$$

Moreover, let N be the least positive integer such that $T \le N\,L$; then we have the explicit bound:

$$c(T) \ge \sin^2\left(\frac{\pi}{2(N+1)}\right). \tag{15}$$

Proof. Let us introduce the anticipated and retarded shift operators:

$$(S_+ f)(t) := H(T-L-t)f(t+L), \qquad (S_- f)(t) := H(t-L)f(t-L).$$

Instead of the non symmetric operator A, we shall consider its symmetric part

$$A_s = \frac{A+A^*}{2},$$

where the asterisk denotes the adjoint of an operator. For every $\varphi \in L^2(\Sigma_T)$, we have

$$a_{\mathcal{E}}(\varphi,\varphi) =< A_s\varphi, \varphi >_{L^2(\Sigma_T)} . \tag{16}$$

Denoting by $\mathcal{R}$ the reflection matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, from a straightforward calculation we obtain

$$A^*\varphi(t) = \frac{1}{2}\big[\varphi(t) + S_+\mathcal{R}\varphi\,(t)\big].$$

Therefore the symmetric part of A has the following expression:

$$2A_s\varphi(t) = \varphi(t) + \frac{1}{2}\,(S_- + S_+)\,\mathcal{R}\varphi(t)\,, \qquad t \in [0,T]. \tag{17}$$

There is a clear analogy between the above formulation of the operator A_s and the usual second order finite differences scheme. In what follows we proceed to make clear this analogy and eventually we shall reduce the action of A_s to that of a matrix operator closely connected to finite differences. In order to do this, for the least integer N such that $T \le NL$, we subdivide the interval $[0, NL]$ into N subintervals of equal length L. Then we reformulate (17) in the following equivalent way:

$$2(S_+^k A_s\varphi)(t) = S_+^k\varphi(t) + \frac{1}{2}\,(S_- + S_+)\,\mathcal{R}S_+^k\varphi(t), \quad t \in [0,L]\,, \quad k = 0,1,\dots,N-1. \tag{18}$$

Let us define, for $k = 0, 1, ...N-1$,

$$\Phi_k(t) := (\Phi_k(0,t), \Phi_k(L,t))^\top := \mathcal{R}^k S_+^k\varphi(t)\,, \qquad t \in [0,L],$$

that is

$$\Phi(0,t) = (\varphi(0,t), S_+\varphi(L,t), S_+^2\varphi(0,t), \dots, S_+^{N-1}\varphi(m_1 * L, t)), \qquad t \in [0,L]\,,$$
$$\Phi(L,t) = (\varphi(L,t), S_+\varphi(0,t), S_+^2\varphi(L,t), \dots, S_+^{N-1}\varphi(m_2 * L, t)), \qquad t \in [0,L]\,,$$

where: $m_1 = [(N-1) - 2\lfloor\frac{N-1}{2}\rfloor]$ and $m_2 = m_1 + (-1)^{N-1}$. Similarly, for the right-hand side in (18), we set

$$\Psi_k(t) := (\Psi_k(0,t), \Psi_k(L,t))^\top := 2(\mathcal{R}^k S_+^k A_s\varphi)(t).$$

Finally, we observe that multiplying both sides of formula (18) by $\mathcal{R}^k$, we obtain $(\Phi_{-1}(t) \equiv \Phi_N(t) \equiv 0)$:

$$\Psi_k(t) = \frac{1}{2}\Phi_{k-1}(t) + \Phi_k(t) + \frac{1}{2}\Phi_{k+1}(t), \quad t \in [0,L], \quad k = 0,1,\dots N-1.$$

Thus each component of $\Psi(t)$ can be expressed as

$$\Psi(0,t) = \mathcal{F}\,\Phi(0,t), \qquad \Psi(L,t) = \mathcal{F}\,\Phi(L,t),$$

where $\mathcal{F} = \text{tridiag}[\frac{1}{2}, 1, \frac{1}{2}]$ is a tridiagonal matrix of order N. Note that the only difference between the matrix $\mathcal{F}$ and the usual finite difference matrix is the sign of the $\frac{1}{2}$'s. In fact, the two matrices are similar through the diagonal matrix which alternates 1 and -1 along the principal diagonal. It is well known that the $N \times N$ finite difference matrix is positive definite and its spectrum is given by the N eigenvalues

$$\omega_k^2 = 2\sin^2\left(\frac{k\pi}{2(N+1)}\right), \qquad k = 1, ..., N\,.$$

Now, having set $\Phi(t) = (\Phi(0,t),\ \Phi(L,t))^\top$, the conclusion follows from the following identity

$$\int_0^T |\varphi(t)|^2\,dt = \sum_{k=0}^{N-1}\int_0^L |\mathcal{R}^k S_+^k \varphi(t)|^2\,dt = \sum_{k=0}^{N-1}\int_0^L |\Phi_k(t)|^2\,dt = \int_0^L |\Phi(t)|^2\,dt,$$

and from the inequality

$$\begin{aligned}
2 < A_s\varphi, \varphi >_{L^2(\Sigma_T)} &= \sum_{k=0}^{N-1}\int_0^L (\mathcal{R}^k S_+^k A_s\varphi)(t)\,\cdot\,(\mathcal{R}^k S_+^k \varphi)(t)\,dt \\
&= \sum_{k=0}^{N-1}\int_0^L \Psi_k(t)\,\cdot\,\Phi_k(t)\,dt = \int_0^L \mathcal{F}\Phi(0,t)\,\cdot\,\Phi(0,t)\,dt + \\
&+ \int_0^L \mathcal{F}\Phi(L,t)\,\cdot\,\Phi(L,t)\,dt \geq \omega_1^2\int_0^L |\Phi(t)|^2\,dt = \omega_1^2\,|\varphi|^2_{L^2(\Sigma_T)}\,,
\end{aligned}$$

remembering (16).

■

At this point we can write down the energetic weak problem related to the BIE (8), which admits a unique, stable solution:

given $g \in H^1_{\{0\}}(\Sigma_T)$, *find* $\varphi \in L^2(\Sigma_T)$ *such that*

$$a_{\mathcal{E}}(\varphi, \psi) = < g_t, \psi >_{L^2(\Sigma_T)}, \quad \forall\,\psi \in L^2(\Sigma_T). \tag{19}$$

2.3 Neumann problem

Similar considerations as those developed in the subsection 2.2 can be done for the wave problem with Neumann boundary condition:

$$u_{tt} - u_{xx} = 0, \qquad x \in \mathbb{R}\setminus\{0, L\}, \quad t \in (0, T) \tag{20}$$

$$u(x,0) = u_t(x,0) = 0, \qquad x \in \mathbb{R}\setminus\{0, L\}, \tag{21}$$

$$\frac{\partial u}{\partial \nu}(x,t) = f(x,t), \qquad (x,t) \in \Sigma_T := \{0, L\} \times [0, T]\,, \tag{22}$$

where $f(x,t)$ is a given function. In this case we have the double layer representation of the solution $u(x,t)$ through the unknown retarded potential ψ:

$$u(x,t) = (K\psi)(x,t) = \sum_{y=0,L}\int_{-\infty}^{+\infty}\frac{\partial}{\partial \nu_y}G(x-y, t-\tau)\,\psi(y,\tau)\,d\tau \tag{23}$$

where $G(\cdot\,,\,\cdot)$ is given by (4).
After a straightforward calculation we obtain the more explicit formula

$$u(x,t) = \frac{1}{2}\frac{x}{|x|}\,\psi(0, t-|x|) - \frac{1}{2}\frac{x-L}{|x-L|}\,\psi(L, t-|x|),$$

from which

$$[u](0,t) := u^-(0,t) - u^+(0,t) = \psi(0,t), \qquad [u](L,t) := u^-(L,t) - u^+(L,t) = \psi(L,t).$$

By deriving formula (23), taking the limits as $x \to 0$ and $x \to L$ and using Neumann data $\frac{\partial}{\partial\nu}u(x,t) = f(x,t)$, we obtain the boundary equations for the unknown potential ψ at the endpoints of the interval $(0,L)$:

$$D\psi := \frac{\partial}{\partial\nu}K\psi = f, \tag{24}$$

that is

$$(D\psi)(0,t) = \frac{1}{2}\Big[\psi_t(0,t) - \psi_t(L,t-L)\Big] = f(0,t)$$

$$(D\psi)(L,t) = -\frac{1}{2}\Big[\psi_t(0,t-L) - \psi_t(L,t)\Big] = f(L,t)\,.$$

In order to derive a weak formulation of the equation (24), our starting point is again the energy identity, which for the Neumann boundary condition assumes the form

$$\mathcal{E}(T) = \int_0^T \frac{\partial u}{\partial\nu}\,\cdot\,[u]_t\,dt = \int_0^T D\psi\,\cdot\,\psi_t\,dt.$$

Thus, having defined the bilinear form

$$\tilde{a}_{\mathcal{E}} : H^1_{\{0\}}(\Sigma_T) \times H^1_{\{0\}}(\Sigma_T) \to \mathbb{R}, \qquad \tilde{a}_{\mathcal{E}}(\psi,\phi) :=< D\psi\,,\,\phi_t\,>_{L^2(\Sigma_T)},$$

the coerciveness of $\tilde{a}_{\mathcal{E}}(\cdot\,,\,\cdot)$ follows at once from the observation that

$$\tilde{a}_{\mathcal{E}}(\psi,\psi) = \int_0^T A\tilde{\psi}_t\,\cdot\,\tilde{\psi}_t\,dt$$

where

$$\tilde{\psi}(t) = (\psi(0,t), -\psi(L,t))^\top,$$

and A is the operator defined in (12). Then from Theorem 1 we have the following

Theorem 2 *For every $T>0$ there exists a positive constant $c(T)$ such that*

$$\tilde{a}_{\mathcal{E}}(\psi,\psi) \geq c(T)\;|\psi_t|^2_{L^2(\Sigma_T)}, \qquad \psi \in H^1_{\{0\}}(\Sigma_T).$$

The constant $c(T)$ is bounded from below as in (15).

Since Dirichlet or Neumann boundary conditions are similar for what concerns the analysis of the related weak formulations, from now on we will consider exclusively the wave equation equipped with Dirichlet boundary conditions.

3 Remarks on energy coerciveness

The coerciveness of the quadratic form $a_{\mathcal{E}}(\varphi,\varphi)$, defined in (13), asserts a coerciveness property of the total energy of the solution u to problem (1)-(3). This follows at once remembering (6) and the equality

$$a_{\mathcal{E}}(\varphi,\varphi) = \mathcal{E}(T)\,.$$

Thus, Theorem 1 assures that

$$\mathcal{E}(T) \geq c(T)\left|\left|\left[\frac{\partial u}{\partial\nu}\right]\right|\right|^2_{L^2(\Sigma_T)}. \tag{25}$$

The purpose of this section is to point out a few interesting facts about the different contributions of the internal and external energies to inequality (25), where the external and internal energies are defined respectively as

$$\mathcal{E}_+(t) := \frac{1}{2}\int_{\mathbb{R}\setminus(0,L)} (u_t(x,t)^2 + u_x(x,t)^2)\, dx, \qquad \mathcal{E}_-(t) := \frac{1}{2}\int_0^L (u_t(x,t)^2 + u_x(x,t)^2)\, dx.$$

We shall see that the main contribution to inequality (25) is provided by the external energy. Indeed, for any given time T, one may replace in (25) the global energy $\mathcal{E}(T)$ with $\mathcal{E}_+(T)$, provided a slightly larger coerciveness constant $\tilde{c}(T)$ takes the place of $c(T)$. On the contrary, interactions of reflected waves make the contribution of the internal energy $\mathcal{E}_-(T)$ almost negligible at least for large times $T \gg L$. More precisely, for any T greater than L, we have (see (28) below):

$$\mathcal{E}_-(T) = \frac{1}{4}\int_{T-L}^{T} |\varphi(t)|^2 dt, \tag{26}$$

thus $\mathcal{E}_-(T)$ vanishes provided $\varphi(t) = 0$ in the "small" interval $(T-L, T)$.

Although our arguments rely upon particular features of the one dimensional D'Alembert equation, they could suggest a possible approach even for the much more difficult cases of the 2 or 3 dimensional wave equation. In fact also in the n-dimensional case, at least for non trapping domains, the external energy at a given time is strictly positive and thus may be viewed as a possible coercive quadratic form in the single layer potential variable φ with respect to a suitable norm $\|\cdot\|_W$. Of course, in the n-dimensional case the main open problem is the identification of the functional space W.

We start by observing that the solution u^+ to problem (1)-(3), on each external interval is a simple progressive wave, that is

$$u^+(x,t) = \begin{cases} g(L, t+L-x) & \text{for } x > L, \\ g(0, t+x) & \text{for } x < 0. \end{cases}$$

Thus, at the boundary points $x = 0$, $x = L$, we have

$$\frac{\partial u^+}{\partial \nu} = -g_t = -(V\varphi)_t, \qquad t \in [0,T].$$

On the other hand, from the energy identity on the external domain and remembering the definition of the operator A, we obtain the following identity

$$\mathcal{E}_+(T) = -\int_0^T \frac{\partial u^+}{\partial \nu}\cdot g_t dt = \int_0^T |(V\varphi)_t|^2 dt = < A\varphi, A\varphi >_{L^2(\Sigma_T)} = < A^*A\varphi, \varphi >_{L^2(\Sigma_T)} . \tag{27}$$

Therefore, as we claimed before, $\mathcal{E}_+(T)$ may be viewed as a coercive quadratic form with coerciveness constant $\tilde{c}(T) = 1/\|A^{-1}\|^2$. See also Section 4 where the same form is introduced in the context of variational formulations related to the BIE (8).
Let us consider the internal energy. From a simple computation, we get the formula

$$A^*A\varphi(t) = A_s\varphi(t) - \frac{1}{4}H(t-T+L)\varphi(t),$$

which, owing to (27), yields the following simple expression for $\mathcal{E}_-(T)$:

$$\mathcal{E}_-(T) = \mathcal{E}(T) - \mathcal{E}_+(T) = < (A_s - A^*A)\varphi, \varphi >_{L^2(\Sigma_T)} = \frac{1}{4}\int_0^T H(t-T+L)|\varphi(t)|^2 dt\,, \tag{28}$$

and (26) is proved.
We conclude this section with other two remarks. Even though the internal energy is in general only nonnegative, it still enjoys some coerciveness property if we add to $\mathcal{E}_-(T)$ its integral with respect to time as in the following

Proposition 2 *For every $T > 0$ we have*

$$|\varphi|^2_{L^2(\Sigma_T)} \leq 4\mathcal{E}_-(T) \, + \frac{8}{L}\int_0^T \mathcal{E}_-(t)dt. \tag{29}$$

Proof. Instead of using directly the identity (28), we shall derive (29) from a simple application of the multipliers technique. Let us define the function

$$q(x) = \frac{2}{L}\big(x - \frac{2}{L}\big) \qquad 0 < x < L.$$

and evaluate through integrations by parts the integral

$$\int_0^T \int_0^L \Big(u_{tt}(x,t) - u_{xx}(x,t)\Big)\, q(x)\, u_x(x,t)\, dx\, dt = 0$$

in terms of space and space-time integrals of quadratic forms, either on the boundary or on the interval $(0, L)$. The function u will be assumed regular enough to perform all the integration by parts, the estimate (29) then will follow by a density argument. We have

$$\begin{aligned}
&\int_0^T \int_0^L u_{tt}(x,t) q(x) u_x(x,t)\, dx\, dt = \\
&\int_0^T \int_0^L \big[u_t(x,t) q(x) u_x(x,t)\big]_t\, dx\, dt - \int_0^T \int_0^L q(x) u_t(x,t) u_{xt}(x,t)\, dx\, dt = \\
&\int_0^L q(x) u_t(x,t) u_x(x,t)\, dx\Big|_{t=T} - \frac{1}{2}\int_0^T \int_0^L \big[q(x) u_t^2(x,t)\big]_x\, dx\, dt + \frac{1}{2}\int_0^T \int_0^L q'(x) u_t^2(x,t)\, dx\, dt = \\
&\int_0^L q(x)\, u_t(x,t)\, u_x(x,t)\, dx\Big|_{t=T} - \frac{1}{2}\int_0^T |g_t(t)|^2\, dt + \frac{1}{2}\int_0^T \int_0^L q'(x) u_t^2(x,t)\, dx\, dt.
\end{aligned} \tag{30}$$

On the other hand

$$\begin{aligned}
\int_0^T \int_0^L u_{xx}(x,t) q(x) u_x(x,t)\, dx\, dt &= \frac{1}{2}\int_0^T \int_0^L (q(x) u_x^2(x,t))_x\, dx\, dt - \frac{1}{2}\int_0^T \int_0^L q'(x)\, u_x^2(x,t)\, dx\, dt = \\
&\frac{1}{2}\int_0^T \big(u_x^2(0,t) + u_x^2(L,t)\big)\, dx - \frac{1}{2}\int_0^T \int_0^L q'(x)\, u_x^2(x,t)\, dx\, dt.
\end{aligned} \tag{31}$$

Thus, summing the equalities (30), (31), we get the identity

$$\int_0^L q(x) u_t(x,t)\, u_x\, dx\Big|_{t=T} + \frac{2}{L}\int_0^T \mathcal{E}_-(t)\, dt = \frac{1}{2}\int_0^T |g_t(t)|^2\, dt + \frac{1}{2}\int_0^T \left|\frac{\partial u^-}{\partial \nu}(t)\right|^2\, dt,$$

and taking into account (27),

$$\int_0^L q(x)\, u_t(x,t)\, u_x(x,t)\, dx\Big|_{t=T} + \frac{2}{L}\int_0^T \mathcal{E}_-(t)\, dt = \frac{1}{2}\int_0^T \left(\left|\frac{\partial u^+}{\partial \nu}\right|^2 + \left|\frac{\partial u^-}{\partial \nu}\right|^2\right)\, dt. \tag{32}$$

By applying the Cauchy-Schwarz inequality in the term of (32) containing the product $u_t u_x$, and using the following inequality

$$|\varphi|^2 = \left|\frac{\partial u^-}{\partial \nu} - \frac{\partial u^+}{\partial \nu}\right|^2 \leq 2\left(\left|\frac{\partial u^-}{\partial \nu}\right|^2 + \left|\frac{\partial u^+}{\partial \nu}\right|^2\right),$$

we have proved the inequality (29).

■

Finally, by combining the identity (27), the energy identity (11) and the inequality (29), we obtain an alternative proof of the coerciveness estimate (14) which does not rely on the algebraic features of the operator A_s. In fact, by an application of the Cauchy-Schwarz inequality in (11), we have

$$\mathcal{E}(t) \le \left(\int_0^t |\varphi(\tau)|^2 \, d\tau\right)^{1/2} \left(\int_0^t |g_t(\tau)|^2 \, d\tau\right)^{1/2} \le |\varphi|_{L^2(\Sigma_T)} \, |g_t|_{L^2(\Sigma_T)}.$$

Therefore, since $\mathcal{E}_-(t) \le \mathcal{E}(t)$, we get from (29) and (27)

$$|\varphi|^2_{L^2(\Sigma_T)} \le 4\mathcal{E}_-(T) + \frac{8T}{L} \, |\varphi|_{L^2(\Sigma_T)} \sqrt{\mathcal{E}_+(T)},$$

which yields the following inequality where the internal and external energies play a distinguished role:

$$|\varphi|^2_{L^2(\Sigma_T)} \le 8\mathcal{E}_-(T) + \frac{64T^2}{L^2} \, \mathcal{E}_+(T).$$

Note that the constant $(8 + 64T^2/L^2)^{-1}$ is not optimal, nevertheless as function of the ratio T/L has the same asymptotic behavior of $c(T)$ in (15).

4 Variational formulations for the BIE $V\varphi = g$

A deeper understanding of the physical phenomenon modelled by differential or integral equations might be obtained by a variational formulation, if this can be written down. In fact, variational formulations can be used as a starting point for theoretical analysis about the characterization of the problem solution, or as a starting point for the construction of numerical solution methods. Nevertheless, it is not always simple to give a variational formulation to a mathematical problem. In particular, in absence of symmetry of the problem governing operator, with respect to a given scalar product, it is impossible to construct a relevant variational formulation. This is the case of the weak problems (10) and (19), related to the BIE $V\varphi = g$, which cannot be rewritten as variational problems, since the operators V and $A = \frac{\partial}{\partial t} V$, respectively, are not self-adjoint with respect to the classical L^2 scalar product.

This difficulty can be overcome in two different directions: the first possibility is to change the scalar product choosing a suitable bilinear form in such a way that the given BIE operator V is self-adjoint with respect to the new one. This way of reasoning was followed in [20], where a time-convolutive bilinear form was introduced. A second different strategy retains the L^2 scalar product and suitably changes the given problem.

Condidering the technique proposed in [20], having indicated with $< \cdot, \cdot >_C$ the time convolution product, and having introduced the bilinear form $a_C(\varphi, \psi) : L^2(\Sigma_T) \times L^2(\Sigma_T) \to \mathbb{R}$ defined by

$$a_C(\varphi, \psi) :=< V\varphi, \psi >_C= \int_0^T \psi(0, T-t) V\varphi(0,t) dt + \int_0^T \psi(L, T-t) V\varphi(L,t) dt \, ,$$

it can be proved that

$$a_C(\varphi, \psi) = a_C(\psi, \varphi).$$

In fact, remembering (7) one has:

$$
\begin{aligned}
a_C(\varphi,\psi) &= \frac{1}{2}\int_0^T \psi(0,T-t)\Big[\int_0^t \varphi(0,\tau)d\tau + \int_0^{t-L}\varphi(L,\tau)d\tau\Big]\,dt\,+\\
&\frac{1}{2}\int_0^T \psi(L,T-t)\Big[\int_0^{t-L} \varphi(0,\tau)d\tau + \int_0^{t}\varphi(L,\tau)d\tau\Big]\,dt =\\
&\frac{1}{2}\int_0^T \psi(0,T-t)\Big[\int_0^T H(t-\tau)\varphi(0,\tau)d\tau + \int_0^{T} H(t-L-\tau)\varphi(L,\tau)d\tau\Big]\,dt\,+\\
&\frac{1}{2}\int_0^T \psi(L,T-t)\Big[\int_0^T H(t-L-\tau)\varphi(0,\tau)d\tau + \int_0^{T} H(t-\tau)\varphi(L,\tau)d\tau\Big]\,dt\,.
\end{aligned}
$$

With the changes of variable $\tau = T-s$ and $t = T-\sigma$, we can proceed as follows:

$$
\begin{aligned}
a_C(\varphi,\psi) &= \frac{1}{2}\int_0^T \psi(0,\sigma)\Big[\int_0^T H(s-\sigma)\varphi(0,T-s)ds + \int_0^{T} H(s-L-\sigma)\varphi(L,T-s)ds\Big]\,d\sigma\,+\\
&\frac{1}{2}\int_0^T \psi(L,\sigma)\Big[\int_0^T H(s-L-\sigma)\varphi(0,T-s)ds + \int_0^{T} H(s-\sigma)\varphi(L,T-s)ds\Big]\,d\sigma\,.
\end{aligned}
$$

At last, changing the order of integration, we arrive to:

$$
\begin{aligned}
a_C(\varphi,\psi) &= \frac{1}{2}\int_0^T \varphi(0,T-s)\Big[\int_0^s \psi(0,\sigma)d\sigma + \int_0^{s-L}\psi(L,\sigma)d\sigma\Big]\,ds\,+\\
&\frac{1}{2}\int_0^T \varphi(L,T-s)\Big[\int_0^{s-L} \psi(0,\sigma)d\sigma + \int_0^{s}\psi(L,\sigma)d\sigma\Big]\,ds = a_C(\psi,\varphi)\,.
\end{aligned}
$$

This result allows to write the following classical characterization:

Proposition 3 *A function $\varphi \in L^2(\Sigma_T)$ is the unique solution of the BIE $V\varphi = g$ if and only if it is a stationary point for the quadratic functional*

$$F_C(\psi) = \frac{1}{2}\,a_C(\psi,\psi) - < g,\psi >_C \ . \tag{33}$$

Unfortunately, the convolutive bilinear form is not coercive. As drawback, we have that the associated weak problem

given $g \in H^1_{\{0\}}(\Sigma_T)$, find $\varphi \in L^2(\Sigma_T)$ such that

$$a_C(\varphi,\psi) = < g,\psi >_C\,, \quad \forall\,\psi \in L^2(\Sigma_T)\,, \tag{34}$$

suffers of the same instability phenomena as (10); further, we cannot give any characterization of φ in terms of *global minimum* point for the corresponding quadratic functional (33). This type of characterization can be obtained in different ways, if we follow the second strategy, i.e. that one of maintaining the classical L^2 scalar product and suitably changing the given problem $V\varphi = g$.
Following [22], we have to consider an invertible symmetric operator K in order to solve the equivalent problem:

$$V^*KV\varphi = V^*Kg, \tag{35}$$

and such that the new operator V^*KV is self-adjoint and possibly coercive with respect to L^2 scalar product.
As operator K, one can certainly choose the identity operator, but in this case the discretization of the weak reformulation of (35) suffers of the same instability phenomena already cited. In fact, with this

choice, we obviously have to consider the operator $V : L^2(\Sigma_T) \to L^2(\Sigma_T)$ and of course the adjoint operator $V^* : L^2(\Sigma_T) \to L^2(\Sigma_T)$. The bilinear form

$$< V^*V\varphi, \varphi >_{L^2(\Sigma_T)} = |V\varphi|^2_{L^2(\Sigma_T)} \tag{36}$$

is certainly positive, but it is not coercive in $L^2(\Sigma_T)$. In fact, in this case V is a compact operator, thus the right-hand side of (36) defines a weaker norm with respect to $|\cdot|_{L^2(\Sigma_T)}$. To conclude, we can say that the approximations converge, but with a derivative lost in the Sobolev scale.
If we instead consider the operator $V : L^2(\Sigma_T) \to H^1_{\{0\}}(\Sigma_T)$, i.e. as an isomorphism between Hilbert spaces, and consequently $V^* : [H^1_{\{0\}}(\Sigma_T)]' \to L^2(\Sigma_T)$, where $[H^1_{\{0\}}(\Sigma_T)]'$ is the dual space of $H^1_{\{0\}}(\Sigma_T)$, a considerably better choice is:

$$K : H^1_{\{0\}}(\Sigma_T) \to [H^1_{\{0\}}(\Sigma_T)]', \quad K = \left(\frac{\partial}{\partial t}\right)^* \frac{\partial}{\partial t},$$

that is

$$< K\, f, g > := < f_t, g_t >_{L^2(\Sigma_T)},$$

where $< \cdot, \cdot >$ is the duality product between $H^1_{\{0\}}(\Sigma_T)$ and $[H^1_{\{0\}}(\Sigma_T)]'$; hence having introduced the symmetric bilinear form

$$a_K(\varphi, \psi) = < V^*KV\varphi, \psi >_{L^2(\Sigma_T)},$$

the weak problem related to (35) reads:

given $g \in H^1_{\{0\}}(\Sigma_T)$, find $\varphi \in L^2(\Sigma_T)$ such that

$$a_K(\varphi, \psi) = < V^*Kg, \psi >_{L^2(\Sigma_T)}, \quad \forall\, \psi \in L^2(\Sigma_T). \tag{37}$$

Remembering the definition of the operator A, it holds:

$$\begin{aligned} a_K(\psi, \psi) &= < KV\psi, V\psi > = < (V\psi)_t, (V\psi)_t >_{L^2(\Sigma_T)} \\ &= < A^*A\psi, \psi >_{L^2(\Sigma_T)} = |A\psi|^2_{L^2(\Sigma_T)} \geq \frac{1}{\|A^{-1}\|^2} |\psi|^2_{L^2(\Sigma_T)}, \end{aligned}$$

and the coerciveness of the bilinear form is verified. Therefore we have the classical variational result:

Proposition 4 *A function $\varphi \in L^2(\Sigma_T)$ is the unique solution of the BIE $V\varphi = g$ if and only if it is the global minimum point for the quadratic functional*

$$F_K(\psi) = \frac{1}{2}\, a_K(\psi, \psi) - < V^*Kg, \psi >_{L^2(\Sigma_T)} .$$

On the other side, the general theory in [22] allows to write the following:

Proposition 5 *A function $\varphi \in L^2(\Sigma_T)$ is the unique solution of the BIE $V\varphi = g$ if and only if it is the global minimum point for the quadratic functional*

$$\tilde{F}_K(\psi) = \frac{1}{2} < K(V\psi - g), V\psi - g > = F_K(\psi) + \frac{1}{2} < Kg, g > .$$

Other characterizations of the solution of the BIE $V\varphi = g$ can be obtained considering the associated problem $A\varphi = g_t$, and noting that the operator A can be split as the sum of its symmetric and skew-symmetric parts:

$$A = A_s + A_{ss},$$

where $A_s = \frac{A+A^*}{2}$ and $A_{ss} = \frac{A-A^*}{2}$. The operator A_s is continuous, self-adjoint with respect to the classical L^2 scalar product and also coercive owing to Theorem 1. Then, following [2], we have the following result:

Proposition 6 *A function $\varphi \in L^2(\Sigma_T)$ is the unique solution of $A^* A_s^{-1} A\varphi = A^* A_s^{-1} g_t$ (or equivalently of $A\varphi = g_t$) if and only if it is the global minimum point for the quadratic functional*

$$F_s(\psi) = \frac{1}{2} < A_s^{-1}(A\psi - g_t), A\psi - g_t >_{L^2(\Sigma_T)} =$$

$$\frac{1}{2} < A_s\psi, \psi >_{L^2(\Sigma_T)} + \frac{1}{2} < A_s^{-1}(g_t - A_{ss}\psi), g_t - A_{ss}\psi >_{L^2(\Sigma_T)} - < \psi, g_t >_{L^2(\Sigma_T)} \ .$$

Remark. The functional $F_s(\psi)$ can be written starting directly from the BIE $V\varphi = g$ and following the strategy proposed in [22], with the following choice for the operator K:

$$K = K_s := \left[\frac{\partial}{\partial t}\right]^* A_s^{-1} \frac{\partial}{\partial t} = \left[\frac{\partial}{\partial t}\right]^* \left[\frac{\frac{\partial}{\partial t} V + V^* \left(\frac{\partial}{\partial t}\right)^*}{2}\right]^{-1} \frac{\partial}{\partial t}.$$

Finally, if we consider the Lagrangian $\mathcal{L} : L^2(\Sigma_T) \times L^2(\Sigma_T) \to \mathbb{R}$ defined by

$$\mathcal{L}(\phi, \psi) = \frac{1}{2} < A_s\phi, \phi >_{L^2(\Sigma_T)} - \frac{1}{2} < A_s\psi, \psi >_{L^2(\Sigma_T)} + < \psi - \phi, g_t - A_{ss}\phi >_{L^2(\Sigma_T)}$$

it holds from [2]:

Proposition 7 *There is a unique saddle point $(\hat{\phi}, \hat{\psi})$ of $\mathcal{L}(\cdot\,,\,\cdot)$ in $L^2(\Sigma_T) \times L^2(\Sigma_T)$, with $\hat{\phi} = \hat{\psi} = \varphi$ being the unique solution $A\varphi = g_t$.*

The Lagrangian $\mathcal{L}$ can be seen as the functional of the two-field *extended* variational formulation related to the problem $A\varphi = g_t$, in the spirit of [7]. In fact, the corresponding two-field *extended* self-adjoint problem is the following:

$$\begin{bmatrix} A_s & A_{ss} \\ A_{ss}^* & -A_s \end{bmatrix} \begin{bmatrix} \hat{\phi} \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} g_t \\ -g_t \end{bmatrix}, \tag{38}$$

with solution $\hat{\phi} = \hat{\psi} = \varphi$.

5 Discretization and numerical results

For the discretization phase, we introduce a uniform decomposition of the time interval $[0, T]$ with time step $\Delta\, t = T/n,\ n \in \mathbb{N}^+$, generated by the $n+1$ instants

$$t_k = k\, \Delta\, t, \quad k = 0, \cdots, n.$$

Denoting by $\mathbb{P}_r,\ r \geq 0$, the space of polynomials of degree less than or equal r, we consider the standard finite element space

$$X_{\Delta\, t}^r = \{v_{\Delta\, t} \in L^2(0, T)\ :\ v_{\Delta\, t|[t_k, t_{k+1}]} \in \mathbb{P}_r,\, r \geq 0,\ k = 0, \cdots, n-1\}\,.$$

Then, considering the finite dimensional space $W_{\Delta\, t}^r = X_{\Delta\, t}^r \times X_{\Delta\, t}^r \subset L^2(\Sigma_T)$, we can write down the discrete form of all the previously introduced weak problems (10), (19), (34), (37). For instance, referring to (19) we have:

given $g \in H^1_{\{0\}}(\Sigma_T)$, find $\varphi_{\Delta\, t} \in W_{\Delta\, t}^r$ such that

$$a_{\mathcal{E}}(\varphi_{\Delta\, t}, \psi_{\Delta\, t}) = < g_{\,t}, \psi_{\Delta\, t} >_{L^2(\Sigma_T)}\,, \quad \forall\, \psi_{\Delta\, t} \in W_{\Delta\, t}^r\,; \tag{39}$$

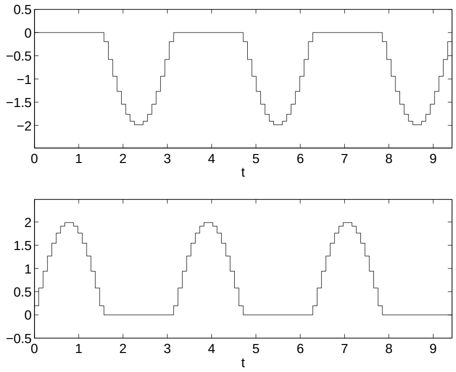

Figure 1: Piece-wise constant approximations $\varphi_{\frac{\pi}{32}}(0,t)$, $\varphi_{\frac{\pi}{32}}(L,t)$ obtained solving $A_{\mathcal{E}}\, x_{\mathcal{E}} = b_{\mathcal{E}}$

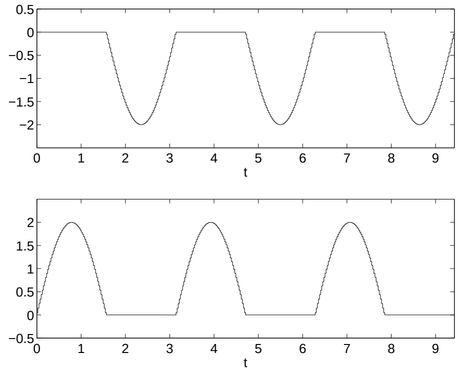

Figure 2: Piece-wise constant approximations $\varphi_{\frac{\pi}{128}}(0,t)$, $\varphi_{\frac{\pi}{128}}(L,t)$ obtained solving $A_{\mathcal{E}}\, x_{\mathcal{E}} = b_{\mathcal{E}}$

in an analogous manner we can write down the discretization of the weak formulation related to the two-field problem (38), in the finite-dimensional space $W^r_{\Delta\, t} \times W^r_{\Delta\, t}$.

Denoting with $\{v_k\}$ a basis for $X^r_{\Delta\, t}$, the unknown function $\varphi_{\Delta\, t}$ can be expressed as:

$$\varphi_{\Delta\, t}(0,t) = \sum_k \varphi^0_k\, v_k(t), \quad \varphi_{\Delta\, t}(L,t) = \sum_k \varphi^L_k\, v_k(t)$$

and the discrete problems can be equivalently written, respectively, as linear systems

$$A_{L^2}\, x_{L^2} = b_{L^2}, \quad A_{\mathcal{E}}\, x_{\mathcal{E}} = b_{\mathcal{E}}, \quad A_C\, x_C = b_C, \quad A_K\, x_K = b_K, \tag{40}$$

in the unknowns the coefficients φ^0_k and φ^L_k. The linear system $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, related to the weak formulation of the extended problem (38), has double dimension with respect to those in (40).

As test problem, let us consider a one dimensional domain of length $L = \frac{\pi}{2}$, subject to the Dirichlet boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = \sin^2(t)\, H(t)\, H(\frac{\pi}{2} - t) + H(t - \frac{\pi}{2}).$$

The observation time interval is $(0, 3\,\pi)$. For the discretization, we have considered time steps of the type $\Delta\, t = \frac{\pi}{2\, p}$, $p \in \mathbb{N}$, such that the time $\frac{\pi}{2}$ required by the wave, travelling with unitary speed, to cover the distance between the two end-points of the domain is a multiple of $\Delta\, t$. Tractions in $x = 0$ and $x = L$ have been approximated by constant or linear shape functions. In Figure 1 we show the numerical solution obtained with $\Delta\, t = \frac{\pi}{32}$ and constant shape functions, solving $A_{\mathcal{E}} x_{\mathcal{E}} = b_{\mathcal{E}}$; if we refine the time mesh choosing $\Delta\, t = \frac{\pi}{128}$ the numerical approximation becomes better, as shown in Figure 2. Note that the same graphs are obtainable solving $A_K x_K = b_K$ or $\mathbf{A}\mathbf{x} = \mathbf{b}$. In Figure 3 instability phenomena arising solving $A_{L^2} x_{L^2} = b_{L^2}$ (or $A_C x_C = b_C$) with the coarse time grid, are presented. This instability still remains even using the smaller time step, as shown in Figure 4, while it disappears if we choose an odd parameter p in the time step $\Delta\, t$ or if we use linear shape functions. The numerical solutions obtained by all the above linear system, with $\Delta\, t = \frac{\pi}{30}$ and constant shape functions, and with $\Delta\, t = \frac{\pi}{32}$ and linear shape functions, are reported in Figures 5, 6 respectively.
If the observation time interval is $(0, 10)$ it natural to choose the time step $\Delta\, t = 0.1$, such that the time $\frac{\pi}{2}$ required by the wave to cover the distance between the two end-points of the domain is not a multiple of it. In Fig. 7 we show the numerical solution obtained solving $A_{\mathcal{E}}\, x_{\mathcal{E}} = b_{\mathcal{E}}$. The same

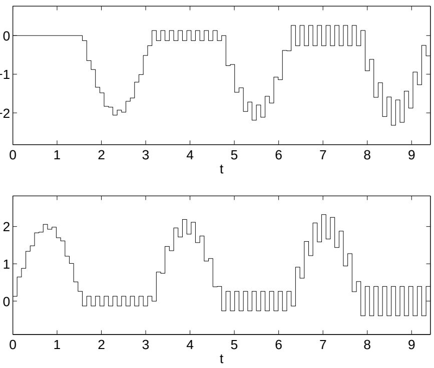

Figure 3: Piece-wise constant unstable approximations $\varphi_{\frac{\pi}{32}}(0,t)$, $\varphi_{\frac{\pi}{32}}(L,t)$ obtained solving $A_{L^2}\, x_{L^2} = b_{L^2}$

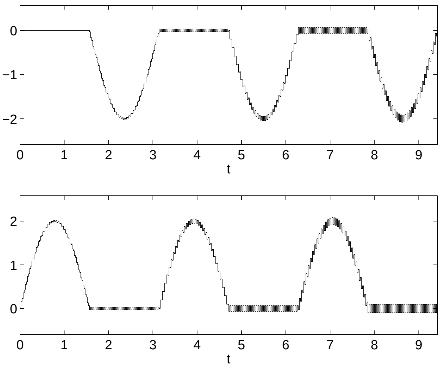

Figure 4: Piece-wise constant unstable approximations $\varphi_{\frac{\pi}{128}}(0,t)$, $\varphi_{\frac{\pi}{128}}(L,t)$ obtained solving $A_{L^2}\, x_{L^2} = b_{L^2}$

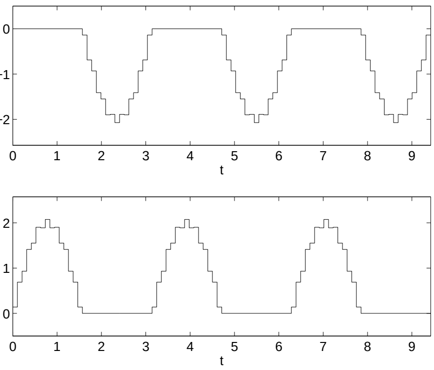

Figure 5: Piece-wise constant approximations $\varphi_{\frac{\pi}{30}}(0,t)$, $\varphi_{\frac{\pi}{30}}(L,t)$ obtained solving solving all linear systems

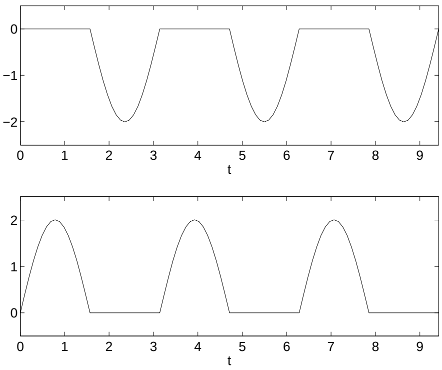

Figure 6: Piece-wise linear approximations $\varphi_{\frac{\pi}{32}}(0,t)$, $\varphi_{\frac{\pi}{32}}(L,t)$ obtained solving all linear systems

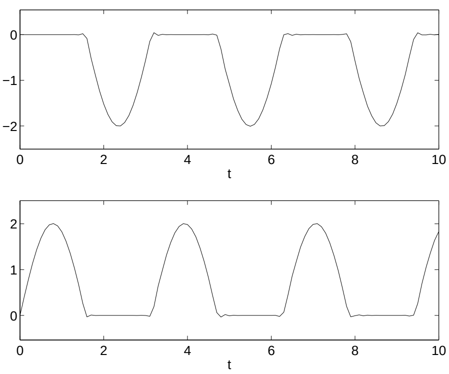

Figure 7: Piece-wise linear approximations $\varphi_{0.1}(0,t)$, $\varphi_{0.1}(L,t)$ obtained solving $A_{\mathcal{E}}\, x_{\mathcal{E}} = b_{\mathcal{E}}$

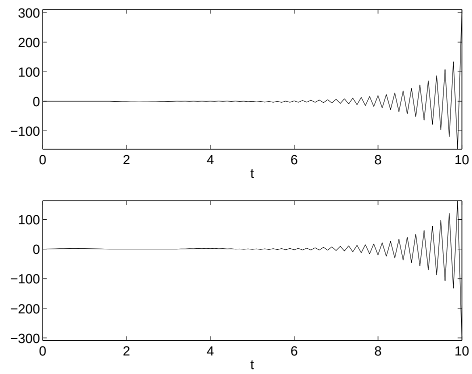

Figure 8: Instability phenomena arising from classical L^2 weak formulation using $\Delta t = 0.1$ and linear shape functions

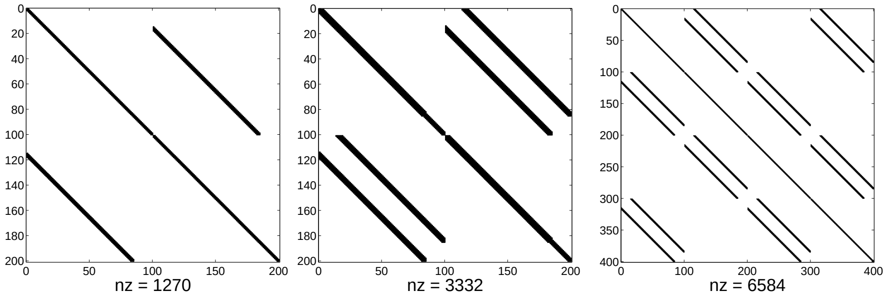

Figure 9: Matrices $A_{\mathcal{E}}$, A_K, $\mathbf{A}$, with $\mu_2(A_{\mathcal{E}}) = \mu_2(\mathbf{A}) \simeq 15.3$ and $\mu_2(A_K) \simeq 234$

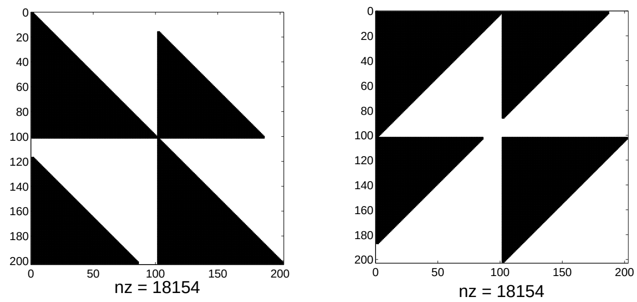

Figure 10: Matrices A_{L^2}, A_C, with $\mu_2(A_{L^2}) = \mu_2(A_C) \simeq 8.6\,10^7$

results can be obtained solving $A_K\, x_K = b_K$ or $\mathbf{A}\mathbf{x} = \mathbf{b}$; in Fig. 8 huge instability phenomena arising solving $A_{L^2}\, x_{L^2} = b_{L^2}$, or $A_C\, x_C = b_C$, are presented. At last, in Figs. 9, 10, the sparsity structure of the above five matrices are reported, together with their spectral condition number.

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Weak and variational formulations for BIEs related to the wave equation

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